 Roll No. ____________

1(CCE-M)6

MATHEMATICS-I

[15]

Time Allowed -3 Hours *Maximum Marks-300*

INSTRUCTIONS

i) *Answers must be written in English.*

ii) *The number of marks carried by each question is indicated at the end of the question.*

iii) *The answer to each question or part there of should begin on a fresh page.*

iv) *Your answer should be precise and coherent.*

v) *The part/parts of the same question must be answered together and should not be interposed between answers to other questions.*

vi) *Candidates should attempt any* ***Five*** *questions.*

vii) *If you encounter any typographical error, please read it as it appears in the text book.*

viii) *Candidates are in their own interest are advised to go through the General Instructions on the back side of the title page of the Answer Script for strict adherence.*

ix) *No Continuation sheets shall be provided to any candidate under any circumstances.*

x) *Candidates shall put a cross (X) on blank pages of Answer Script.*

xi) *No blank page be left in between answer to various questions.*

xii) *No programmable Calculator is allowed.*

xiii) *No stencil (with different markings) is allowed.*

xiv) *In no circumstances help of scribe will be allowed.*

MATHEMATICS-I (12 Questions)

All questions carry same marks (60 each) and each part of every question carry the same weightage

1. Let U and W be subspaces of a vector space V. We define the sum U+W as follows : $U+W=\{u+w : u \in U, w \in W\}$.

   a) Show that U+W is a subspace of V containing both U and W.

   b) Show that span $\{u_1, u_2, ..u_r, w_1, w_2,...w_s\}$= span $\{u_1, u_2,..., u_r\}$ + span $\{w_1, w_2,..., w_s\}$for any vectors u_i's and w_j's t from U and W respectively.

c) If U and W are finite dimensional and if $U \cap W = \{0\}$, then show that dim(U+W) = dim (U) + dim (W).

2. Let $T:V \rightarrow V$ be a linear transformation of a finite dimensional vector space V.

a) Prove that if $U \subset V$ is a vector subspace, then $T(U) = \{T(u): u \in U\}$ is a vector subspace of V.

b) Prove that if T is onto, it sends linearly independent set of vectors to linearly independent set of vectors.

c) Prove that if T is invertible and $U \subseteq V$ then dim T(U) = dim U.

3. Let A and B be m×n matrices.

a) Show that rank (A+B) $\leq$ rank (A) + rank (B)

b) Show that null (A) = null (UA) for any invertible m×m matrix U.

c) Show that dim (null (A) = dim (null (AV)) for any invertible n×n matrix V.

4. A complex matrix S is called skew-Hermitian,if $S^* = -S$

a) Show that $Z - Z^*$ is skew-Hermitian for any square complex matrix Z.

b) If S is skew-Hermitian, then show that S^2 and iS are Hermitian.

c) If S is skew-Hermitian, then show that the eigenvalues of S are purely imaginary ($i\lambda$ *for real* λ).

5. Find the equation (s) of the tangent (s).

a) to the parabola $y^2 = 2ax$ at the vertex and at the ends of Latus rectum.

b) to the hyperbola $x^2 - 2y^2 = 1$ from the point (7,5).

c) to the ellipse $\frac{x^2}{32} + \frac{y^2}{18} = 1$ at the point whose x coordinate is 2.

6. Prove the following :

a) Let $A \subset \mathbb{R}$. We say that a function $\int : A \rightarrow \mathbb{R}$ is Lipschilz on A if there exists L > 0 such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in A$. Show that any Lipschitz function is continuous.

such that $|f(x)| > \frac{|f(c)|}{2}$ for all $x \in (c-\delta, c+\delta) \cap A$.

c) Let $f:[0,2\pi] \to [0,2\pi]$ be continuous such that $f(0) = f(2\pi)$. Show that there exists $x \in [0,2\pi]$ such that $f(x) = f(x+\pi)$.

7. a) Use Lagrange multipliers to find the maxima and minima of the function

$f(x,y) = x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$

b) Find all the stationery points and local maxima and minima of the function

$f(x,y) = e^{x+y}(x^2 + y^2 - xy)$.

8. Let $F(y) = \int_0^\infty \frac{\sin xy}{x(x^2+1)} dx$ *if* $y > 0$

a) show that F satisfies the differential equation $F''(y) - F(y) + \pi/2 = 0$

b) Deduce that $F(y) = \frac{1}{2}\pi(1 - e^{-y})$

c) Deduce that for $y > 0$ *and* $a > 0$ $\int_0^\infty \frac{\cos xy}{x^2 + a^2} dx = \frac{\pi e^{-ay}}{2a}$

9. a) Solve the differential equation $y \cos x + 2xe^y + (\sin x + x^2 e^y - 1) y' = 0$

b) A radioactive substance obeys the equation $y' = ky$ where k<0 and y is the mass of the substance at time t. Suppose that initially, the mass of the substance is y(0) =M>0. At what time does half of the mass remain?

c) The half life of carbon-14 is 5730 years. If one starts with 100 milligrams of carbon-14; how much is left after 6000 years? how long do we have to wait before there is less than 2 milligrams?

10. Consider the initial value problem $y' + \frac{y}{4} = 3 + 2\cos 2t, y(0) = y_0$

a) Find the solution to the homogenous equation of this problem.

b) Find the solution of this initial value problem and describe its behaviour for large t.

c) Determine the value of t for which the solution first intersects the line y=12.

11. Let u and v be vector fields, ϕ is a scalar function and T is a tensor field. Show that

a) $curl\,(grad\,v)^T = grad\,(curl\,v)$

b) $curl\ curl\,v = grad\,(div\,v) - grad^2\,v$

c) $curl\,(\phi v) = \phi\,curl\,v + (\nabla\phi)\times v$

d) $div(u\otimes v) = (div\,v)u + (grad\,u)v$.

12. a) The tallest spot on Earth is Mt. Everest, which is 8857 m above sea level. If the radius of the Earth to sea level is 6369 km, how much does the magnitude of g change between sea level and the top of Mt. Everest?

b) The value of g at the surface of the earth is 9.78 N/kg, and on the surface of Venus the magnitude of g is 8.6 N/kg. An astronaut has a mass of 70 kg on the surface of the earth. What will her weight be on the surface of Venus?

c) A person in a kayak starts paddling, and it accelerates from 0 to 0.9 miles/hour in a distance of 0.9 km. If the combined mass of the person and the kayak is 80 kg, what is the magnitude of the net force acting on the kayak?
